# SOME RESULTS ON DISCREPANCIES BETWEEN METRIC DIMENSION AND PARTITION DIMENSION OF A GRAPH*

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*ABSTRACT. In this paper some infinite regular graphs generated by tilings of the plane by infinite hexagonal grid are considered. It is proved that these graphs have discrepancies between their metric dimension and partition dimension. Also, it is proved that for every $n \geq 2$ there exist finite induced subgraphs of these graphs having metric dimension equal to $n$ as well as infinite induced subgraphs with metric dimension equal to three. It is natural to ask for a characterization of graphs having discrepancies between their metric dimension and partition dimension.*

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## 1 INTRODUCTION AND PRELIMINARY RESULTS

If $G$ is a connected graph, the *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ...., w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The *representation* $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v,w_1), d(v,w_2), ....., d(v,w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a *resolving set* for $G$ [1]. A resolving set of minimum cardinality is called a *basis* for $G$ and this cardinality is the *metric dimension* of $G$, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1,3, 8-13]).

For a given ordered set of vertices $W = \{w_1, w_2, ...., w_k\}$ of a graph $G$, the $i$-th component of $r(v|W)$ is $0$ if and only if $v = w_i$. Thus, to show that $W$ is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

A useful property in finding $dim(G)$ is the following:

**Lemma:** [13] Let $W$ be a resolving set for a connected graph $(G)$ and $u,v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.

Another kind of dimension of a connected graph, called partition dimension was introduced in [4, 5] as follows: For a subset $S \subset V(G)$ and a vertex $v$ of a connected graph $G$, the distance $d(v,S)$ between $v$ and $S$ is defined as usually by $d(v,S) = \min\{d(v,x) : x \in S\}$. If $\Pi = (S_1, S_2, ...., S_k)$ is an ordered $k$-partition of $V(G)$, the representation of $v$ with respect to $\Pi$ is the $k$-tuple $r(v|\Pi) = (d(v,S_1), d(v,S_2), ....., d(v,S_k))$. If the $k$-tuples $r(v|\Pi)$ for $v \in V(G)$ are all distinct, then the partition $\Pi$ is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of $G$ and is denoted by $pd(G)$. Let $\Pi = \{S_1, S_2, ...., S_k\}$ be an ordered partition of $V(G)$. If $u \in S_i, v \in S_j$ where $1 \leq i, j \leq k$ and $i \neq j$, then $r(u|\Pi) \neq r(v|\Pi)$ since $d(u,S_i) = 0$ but $d(v,S_i) \neq 0$. Thus, when determining whether a given partition $\Pi$ of $V(G)$ is a resolving partition for $V(G)$, we need only to verify if the vertices of $G$ belonging to the same class of $\Pi$ have distinct representations with respect to $\Pi$. When $d(u,S_i) \neq d(v,S_i)$ we shall say that the class $S_i$ distinguishes vertices $u$ and $v$. Another useful property in determining $pd(G)$ is the following lemma [5].

**Lemma:** Let $\Pi$ be a resolving partition of $V(G)$ and $u,v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $u$ and $v$ belong to different classes of $\Pi$.

It is natural to think that the partition dimension and metric dimension are related; in [4] it was shown that for any nontrivial connected graph $G$ we have $pd(G) \leq dim(G) + 1$.

However, the partition dimension may be much smaller than the metric dimension.

These concepts have some applications in chemistry for representing chemical compounds [3,8] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [9].

Let $(i, j)$ and $(i', j')$ be two points with integral coordinates in $Z^2$. It is well known that the following definitions yield metrics for $\mathrm{Z}^2$: $d_4((i,j),(i',j')) = |i-i'|+|j-j'|$ (city block distance) and $d_8((i,j),(i',j')) = \max(|i-i'|,|j-j'|)$ (chessboard distance). The indices 4 and 8 are appropriate because they represent the number of points at distance one(the neighbors) from a given point with respect to these two metrics. These two metrics on $\mathrm{Z}^2$ generate two infinite graphs $(\mathrm{Z}^2,\mathrm{E}_4)$ and $(\mathrm{Z}^2,\mathrm{E}_8)$ having the same vertex set $\mathrm{Z}^2$ and the set of edges consisting of all pairs of vertices whose city block and chessboard distances are $1$. $(\mathrm{Z}^2,\mathrm{E}_4)$ is a planar $4$-regular graph whose regions are unit squares and it is also known as the square lattice graph. $(\mathrm{Z}^2,\mathrm{E}_8)$ is $8$-regular and can be obtained from $(\mathrm{Z}^2,\mathrm{E}_4)$ by drawing all diagonals of unit squares. In [9] it was proved that these two graphs have no finite metric bases and for any natural number $n \ge 3$, there exist induced subgraphs of $(\mathrm{Z}^2,\mathrm{E}_4)$ and $(\mathrm{Z}^2,\mathrm{E}_8)$, respectively having metric dimension equal to $n$ and partition dimension equal to three. Also, in [12] it was shown that $pd(\mathrm{Z}^2,\mathrm{E}_4)=3$ and $pd(\mathrm{Z}^2,\mathrm{E}_8)=4$. In this way, Tomescu was the first who proved that the partition dimension of a graph may be much smaller than the metric dimension. He called this phenomena for connected graphs when the metric dimension of a graph is infinite but its partition dimension is finite, the discrepancy between metric dimension and partition dimension. In [13], it was also proved that that infinite hexagonal grid and some infinite regular graphs generated by tiling of the plane by triangular lattice have also discrepancy between their metric dimension and partition dimension. It is natural to ask for a characterization of graphs having discrepancies between their metric dimension and partition dimension.

In this paper, we extend this study by considering some infinite regular graphs generated by tiling of the plane by an infinite hexagonal grid. We prove that these graphs have discrepancies between their metric dimension and partition dimension. Also there exist finite induced subgraphs of these graphs having metric dimension equal to $n$ as well as infinite induced subgraphs with finite and constant metric dimension.

## *2 SOME INFINITE REGULAR GRAPHS GENRATED BY INFINITE HEXAGONAL GRID*

In what follows we shall consider some infinite regular graphs generated by tilings of the plane by infinite hexagonal grid.

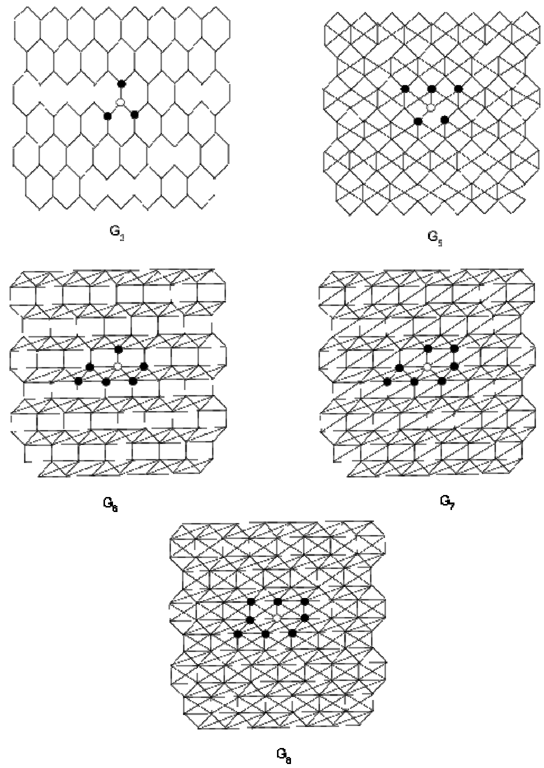

Figure 1: The infinite graphs $G_3, G_5, G_6, G_7$ and $G_8$

The graph $G_3$ is a planar $3$-regular infinite graph whose regions are regular hexagons of unit side. $V(G_3)$ consists of the vertices of these hexagons, two vertices being adjacent if they are the extremities of a unit side of a hexagon in the tiling and it is known as *infinite hexagonal grid*. The graph $G_5$ is a $5$-regular infinite graph which can be obtained from $G_3$ by drawing its two diagonals except the horizontal and vertical diagonals of regular hexagons. A $6$-regular infinite graph denoted by $G_6$ can be obtained from $G_3$ by joining those vertices by an edge that are extremities of horizontal diagonals of regular hexagons and by drawing the other diagonal of the rhombuses appearing in the graph. This graph induces the rectangles and rhombuses (having both diagonals). If we draw one diagonal (having slope $\frac{1}{\sqrt{3}}$) appearing in the infinite graph $G_6$, we get a $7$-regular infinite graph denoted by $G_7$. The graph $G_7$ induces the rectangles with one diagonal and rhombuses having both diagonals. Similarly if we draw both diagonals of the rectangles (having slope $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$), we get an $8$-regular infinite graph denoted by $G_8$ (see Fig. $1$). The indices $3,5,6,7$ and $8$ are appropriate because they represent the number of vertices (the neighbors) at distance $1$ from a given vertex.

## 3 MAIN RESULTS ON DISCREPANCIES BETWEEN METRIC DIMENSION AND PARTITION IMENSION

In this section, we prove that the graphs $G_3, G_5, G_6, G_7$ and $G_8$ have discrepancy between their metric dimension and partition dimension. In the next theorem, we show that the graphs $G_3, G_5, G_6, G_7$ and $G_8$ have no finite metric bases. **THEOREM 1.** The graphs $G_3, G_5, G_6, G_7$ and $G_8$ have no finite metric basis, i.e., $dim(G_3) = dim(G_5) = dim(G_6) = dim(G_7) = dim(G_8) = \infty$.

**Proof:** Figs. $2$ represent two vertices $x, y$ in $G_3, G_5, G_6, G_7$ and $G_8$ having their Euclidean distances equal to $1, \sqrt{3}$ and to $\sqrt{2}$, respectively and subgraphs $G_i(x, y)(i = 3,5,6,7,8)$ consisting of vertices $z$ such that $d(x,z) = d(z, y)$. Suppose that $G_5$ has a finite metric basis $S$. We can find two vertices $x, y$ and a subset $T \subset G_5(x, y)$ consisting of all vertices $z \in G_5(x, y)$ such that $d(z, x) = d(z, y) \le k$ for $k$ large enough, such that $S \subset T$. This implies that $d(x,z) = d(y,z)$ for all $z \in S$, a contradiction. The proof is similar for other infinite graphs.

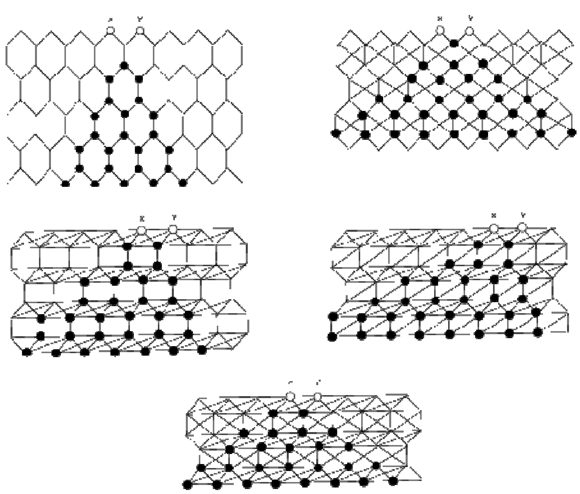

Figure 2: Subgraph of vertices having equal distances to $x$ and $y$

In the next theorem, we determine the exact value for partition dimension of $G_3, G_5$ and $G_6$ which shows that partition dimension of these graphs is finite.

**THEOREM 2.** We have $pd(G_3) = pd(G_5) = pd(G_6) = 3$.

**Proof:** In [3] it was shown that $pd(G) = 2$ if and only if $G$ is a path and this property also holds for infinite graphs. It follows that $pd(G_3) \ge 3, pd(G_5) \ge 3$ and $pd(G_6) \ge 3$.

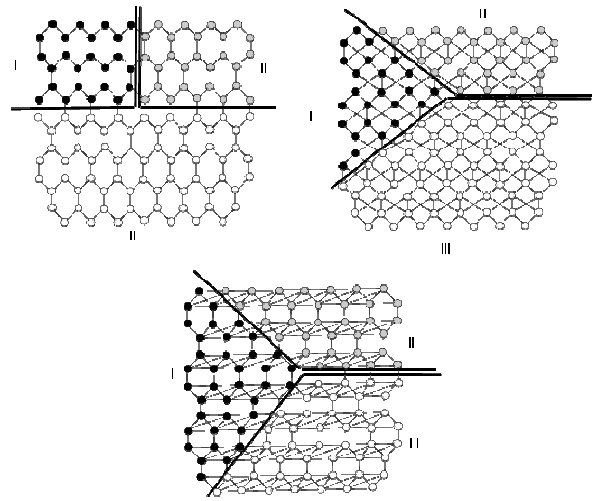


Figure 3: Resolving $3$-partitions of $V(G_3), V(G_5)$ and $V(G_6)$

Fig. $3$ provides resolving $3$-partitions of $G_3, G_5$ and $G_6$, respectively. It follows that $pd(G_3) = pd(G_5) = pd(G_6) = 3$.

The problem of determining partition dimension of $G_7$ and $G_8$ is much more difficult. We are only able to find some bounds in the next theorem but these bounds are enough to prove that these graphs have discrepancies between their metric dimension and partition dimension.

**THEOREM 3.** We have $3 \le pd(G_7) \le 4$ and $3 \le pd(G_8) \le 5$.

**Proof:** The same argument used in Theorem $2$ implies that $pd(G_7) \ge 3$ and $pd(G_8) \ge 3$. On the other hand, Fig. $4$ provides a resolving $4$-partition of $V(G_7)$ and a resolving $5$-partition of $V(G_8)$, which completes the proof.

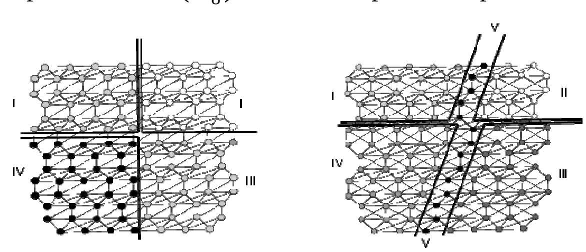


Figure 4: Resolving $4$-partitions of $V(G_{10})$ and resolving $5$-partition of $V(G_{10})$

The metric dimension of some induced subgraphs of $G_3$ has been studied in [13] and it was proved that there are induced subgraphs of these graphs having metric dimension equal to $n$ as well as there are infinite induced subgraphs with metric dimension equal to three.

Fig. $5$ represents some induced subgraphs of $G_5, G_6, G_7$ and $G_8$: $F_n$ is an induced subgraph of $G_5$ and contains $n$ hexagons with two of its diagonals except the horizontal and vertical diagonals. $H_1$ (two-way infinite ladder) and $H_2$ (two-way infinite triangular ladder) are infinite induced subgraphs of $G_6$ and $G_7$, respectively consisting of triangles(rectangles) whereas $H_3$ contains rectangles with both diagonals, and is infinite induced subgraph of $G_8$.

### *4 INFINITE REGULAR GRAPHS WITH FINITE METRIC DIMENSION*

A natural starting point for studying the metric dimension of infinite graphs is to investigate the finiteness of this invariant. It is not difficult to realize that infinite graphs may have finite or infinite metric dimension. In fact, for every $k \geq 0$ there exist infinite graphs with metric dimension $k$. Caceres et al. [2] proved the following result for infinite graphs.

- The metric dimension of a graph $G$ is 1 if, and only if, $G$ is either a finite path or the one-way infinite path.
- The infinite comb graph $B_\infty$ has infinite metric dimension.
- An infinite tree has finite metric dimension if and only if the set of vertices of degree at least three is finite.

They also obtained some result on cartesian product of infinite graphs. For detail see [2]. Tomescu proved in [12] that infinite regular graphs $(\mathsf{Z}^2, \mathsf{E}_4)$ and $(\mathsf{Z}^2, \mathsf{E}_8)$ have infinite metric dimension and in [13] it was also shown that graphs generated by tiling of the plane by regular triangles and hexagons have no finite metric bases.

In the next theorem we give the example of a couple of infinite regular graphs having finite metric dimension which shows that not every infinite regular graph has infinite metric dimension.

**THEOREM 4.** We have $dim(H_1) = dim(H_2) = 3$ but $dim(H_3) = \infty$. **Proof:** For $H_1$ (two-way infinite ladder) and $H_2$ (two-way infinite triangular ladder), the vertices which are at the same distance apart from $B$ are distinguished by $A$ and $C$, respectively. In the case of $H_3$ (two-way infinite diagonal ladder), each rectangle inducing a complete graph $K_4$ has two pairs of vertices $\{x, y\}$ such that $d(x,z) = d(y,z)$ for every $z \in V(H_3) \setminus \{x, y\}$. By Lemma $1$ this implies that $dim(H_3) = \infty$.

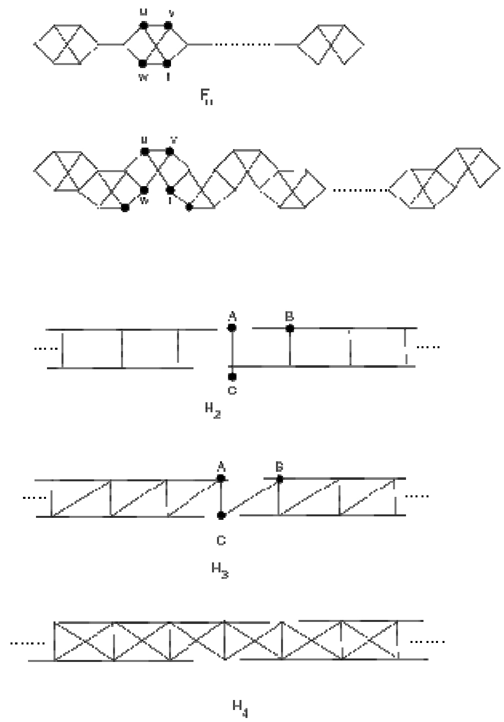

Figure 5: Some induced subgraphs of $G_i (5 \leq i \leq 8)$

**THEOREM 5.** For every $n \geq 2$ we have $dim(F_n) = 2n$ and $F_n$ has $n!4^n$ metric bases. **Proof:** The vertices $u$ and $w$ of $F_n$ have equal distances to all vertices of $F_n$. Similarly vertices $v$ and $t$ have equal distances to all vertices of $F_n$. It follows that at least one of $u, v$ and one of $w, t$ from each hexagon of $F_n$ must belong to any resolving set of $F_n$ implying that $dim(F_n) \geq 2n$.

On the other hand, by choosing two vertices of degree three other than end vertices in each hexagon of $F_n$ ( in $4^n$ ways), the set of these vertices forms a metric basis of $F_n$. These vertices can be ordered in $n!$ ways and the result follows.

### *5 CONCLUSION*

In this paper, some infinite regular graphs generated by tilings of the plane by infinite hexagonal grid are considered. These graphs have no finite metric bases but their partition dimension is finite and is evaluated in some cases. It is natural to ask for a characterization of graphs having discrepancies between their metric dimension and partition dimension.

Also, it is proved that for every $n \geq 2$ there exist finite induced subgraphs of these graphs having metric dimension equal to $n$ as well as infinite induced subgraphs with metric dimension equal to three. We close this section by raising some questions that naturally arise from the text.

**Open Problem 1**: *Is it the case that every infinite regular graph generated by tiling of the plane by infinite hexagonal grid will always have the discrepancies between their metric dimension and partition dimension?*

**Open Problem 2**: *Find the exact value of partition dimension for* $G_7$ *and* $G_8$.

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